

# Generalized Maximum Dynamic Contraflow on Lossy Network

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## Abstract

Generalized maximum dynamic flow, earliest arrival flow and contraflow problems have been extensively discussed as core problems in the research of evacuation planning. In this work, we formulate and justify the solution procedure of two problems. First, we present the generalized maximum dynamic contraflow problem where a maximum amount of evacuees can be sent to a safe destination from the dangerous place within a given time horizon with minimum loss by allowing reversing the direction of the roads. Second, the generalized earliest arrival contraflow is formulated where as many evacuees as possible should be sent in every time period to a safe destination from dangerous place through highest gain path allowing reversing the direction of the roads. A pseudo-polynomial time algorithm for both problems will be presented on a single source and a single sink lossy network having capacities, transit times and gain factors on the arcs.

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**Keywords:** Evacuation planning; lossy network; generalized maximum contraflow; generalized earliest arrival contraflow.

## 1. Introduction

Throughout the world, evacuation plans are developed to react to different emergency problems due to both natural and man-made disasters. In these plans, the mathematical model of the transportation network plays an important role during the evacuation process and removing traffic jam is one of the main issues. There are different mathematical models to deal with evacuation planning problems covering discrete as well as continuous time problems, macroscopic and microscopic behavioral considerations, optimization and simulation techniques and approximate solution techniques including heuristics.

The dynamic network flow model is one of major models to deal with the transportation issue. In the dynamic network flow problem, the flow requires a certain amount of time to travel through each street represented by an arc of the network. In order to deal with the evacuation planning problems, different dynamic flow problems like the maximum dynamic flow (MDF) problem, the earliest arrival flow (EAF) problem as well as universal maximum flow (UMF) problem, the quickest flow problem or the transshipment problems are considered in literature. The dynamic flow problems vary

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corresponding to the number of sources and sinks, to the dependencies on characteristic of the arcs and nodes, for example, constant, time- dependent or flow dependent capacities or transit times on arcs or as well as additional constraints or variables. The time steps in the dynamic network flow problems may be discrete or continuous.

In this paper, we use discrete time dynamic network flow problems with constant travel times and transportation capacities, i.e. data which are identical for every time period. This assumption yields lower bounds for the real evacuation time. An overview over mathematical modeling of evacuation problems on building evacuation with the variations of discrete time dynamic network flow problems has been presented by Hamacher and Tjandra [14].

In evacuation planning, there is not sure that flow conservation satisfies for all time periods in the evacuation scenarios. In the dynamic network flow problems, the flow conservation is satisfied for all nodes leaving source and sink. But, in real life scenario, the flow conservation may not be possible always. Therefore, another important part of the transportation network flow problem is the physically transformation of flow which may be lesser or greater amount of flow. Such kinds of network flow problems are called generalized flow problems which are the natural generalization of the dynamic flow problems. In literature, the generalized flow is introduced for the production planning as an important production tool. In this problem, flow is not necessarily conserved on every arc but may be physically transformed due to leakage, evaporation, breeding, theft or interest rates [17]. The generalized flow model can be used for evacuation planning problems where the flow conservation may not be satisfied, for instance due to the death of evacuees or hold-over in arcs [13]. The generalized flow problem was presented by Dantzig [6] giving a linear programming formulation and can thus be solve in polynomial time. The combinatorial polynomial time algorithm proposed by Goldberg et al. [11] for finding generalized maximum flows has been improved later by Radzik ([22], [23]). Most of the polynomial time algorithms developed in the literature have been described by Wayne ([29], [30]) for the generalized flow problem. In the generalized maximum flows, each arc has an additional gain or loss factor. If the negative logarithm of a gain factor is used as the cost, the generalized maximum flow (GMF) algorithms will be similar to minimum cost flow algorithms. But - in contrast to to the minimum cost flow problem [28] - it is an open problem whether a strongly polynomial time algorithm exists for the GMF problem.

The MDF problem was presented by Ford and Fulkerson [8] with the goal to send a maximum amount of flow (evacuees) from a source (dangerous area) to a sink (safe destination) in a given time horizon assuming the transit times and inflow capacities of arcs are constants. They solved the problem optimally by presenting the temporally repeated flow technique. The latter is based on minimum cost flows using travel time as cost function and a chain decomposition into source-sink paths each of which will reach the sink before the observed time horizon will end. They also introduced the time expanded network and demonstrated that the general MDF is equivalent to the static flow in the time expanded network.

Gale [10] presented the EAF problem which is also called UMF problem as an extension of the MDF problem with the additional property that the flow reaching the sink is maximal in every considered time period  $\theta, 0 \leq \theta \leq T$ . A solution obtained by the EAF problem has the advantage that the user does not have to predict a time period  $T$  in which the evacuation has to finish. For the EAF problem, Hoppe [15] analyzed the exact algorithms of ([19], [31]) and presented polynomial algorithms using chain decomposable flows. For the case of time dependent transit times and capacities, Tjandra [27] propose an algorithm for the EAF problem which is polynomial in the time horizon  $T$  and the maximum capacity. Baumann [2] and Baumann and Skutella [3] gave an algorithm for multiple sources and a single sink EAF problem which is polynomial time in the input and output size of the occurrence. In their solution, they recursively construct an EAF pattern and convert it into EAF. For the EAF problem on series-parallel graphs, a polynomial time algorithm was presented by Steiner [26] and Ruzika et al. [25] which is based on a result of Bein et al. [4] for solving minimum cost flow problem on series-parallel graphs.

Burkard et al. [5] proposed a strongly polynomial algorithm to solve the quickest flow problem for a single source and single sink network reducing it to a sequence of MDF problems. Their algorithm is based on parametric search with respect to the time horizon, e.g. binary search. For the general quickest flow problem and the lexicographic maximum dynamic flow problem, Hoppe and Tardos [16] presented polynomial time algorithms. The lexicographic maximum dynamic flow problem maximizes the amounts of flow leaving the sources in a specified order, i.e. the sources are ordered from high to low priority. They introduce generalized temporally repeated flows where flow is allowed to travel in the opposite direction of an edge with negative travel time.

Later, Gross and Skutella [13] introduced a generalized maximum dynamic flow (GMDF) model in networks where each arc contains both gain factors and transit times. In general, solving simultaneously both problems - generalized flows and maximum dynamic flows with gain factor and transit times on every arc - is NP-hard. A pseudo-polynomial time algorithm has been presented for GMDF [13] on lossy network in which the loss rate per time unit is identical on all arcs. They also proved the existence of a generalized earliest arrival flow (GEAF) on lossy network.

In evacuation problems due to various kinds of disaster like hurricane, tsunami, flooding, earthquakes etc., contraflow has been considered as a potential remedy to tackle the problem of traffic jam by increasing the outbound evacuation route capacity. Contraflow for the transportation network also maximizes the number of evacuees reaching safe destinations for each time period. In the literature, there are only few optimization techniques presented to deal with contraflow problem. Kim et al. [18] gave first integer programming formulations of the problem. The maximum static contraflow (MCF) problem for general graphs and the maximum dynamic contraflow problem (MDCF) for networks with a single source and a single sink have been considered by Rebennack et al. [24] and Arulselvan [1]. They presented the first polynomial time algorithms for both

problems. In their papers, it is shown that the quickest contraflow problem can be solved in strongly polynomial time complexity for a single source and a single sink, but the problem in the multiple sources and multiple sinks are NP-hard. Dhamala and Pyakurel [7] presented a polynomial time algorithm for the earliest arrival contraflow (EACF) problem on a single source and a single sink series parallel graph.

In this paper, we introduce the generalized maximum dynamic contraflow (GMDCF) problem and present a pseudo-polynomial time algorithm for this problem for single source and single sink lossy network. Our algorithm is based on the algorithm of Rebennack et al. [24] and Arulselvan [1] for computing the MDCF on general graphs and the algorithm of Gross and Skutella [13] for computing a GMDF on lossy networks with both gain factors and transit times on the arcs. The presented algorithm also gives the generalized earliest arrival contraflow (GEACF) solution on lossy network. To the best of our knowledge, this is the first pseudo-polynomial time algorithm to find the GMDCF as well as the GEACF.

The paper is organized as follows. In Section 2, we summarize some well-known results and notions used in the paper. In Section 3, we establish our main result on the GMDCF and the GEACF on lossy network. Concluding remarks are given in Section 4.

## 2. Preliminaries

Let  $G = (V, E)$  be a directed graph with set of nodes  $V$ , set of arcs  $E$ , capacities  $u_e \in \mathbb{R}_0^+$ , transit times  $\tau_e \in \mathbb{Z}^+$  and gain factors  $\gamma_e \in \mathbb{R}^+$  on the arcs  $e \in E$ . The capacity  $u_e \in \mathbb{R}_0^+$  denotes the maximum amount of flow which may enter the arc  $e \in E$  per time period and the transit time  $\tau_e \in \mathbb{Z}^+$  gives the time needed to travel arc  $e \in E$ . For each time unit of flow entering the tail of an arc  $e \in E$  at time  $\theta$ , only  $\gamma_e$  units leave the arc at its head at time  $\theta + \tau_e$ . We assume that, we are given a single source  $S^+ \in V$  and a single sink  $S^- \in V$  and  $S^+ \neq S^-$ . The source node denotes the dangerous place where evacuees are situated and the sink node is the safe destination having enough capacity for the evacuees from the source. So in general, it is assumed that there are one-way paths from the source to the sink during the evacuation in a standard transportation network. Moreover, a finite time horizon  $T \in \mathbb{Z}^+$  is given for the flow to travel through the network with discrete time steps. Hence, combining these data, a transportation network is defined by  $(G, u, \tau, \gamma, S^+, S^-, T)$ .

Let  $\delta^+(v)$  and  $\delta^-(v)$  denote the set of outgoing and incoming arcs for node  $v \in V$ , respectively. A generalized dynamic flow (GDF)  $f: E \times \{0, 1, \dots, T-1\} \rightarrow \mathbb{R}_0^+$  in the transportation network  $(G, u, \tau, \gamma, S^+, S^-, T)$  is a mapping that gives flow values  $f_{e,\theta} \in [0, u_e]$  to every arc  $e \in E$  at every discrete time point  $\theta \in \{0, 1, \dots, T-1\}$  with respect to the generalized flow conservation constraints as follows (see [13]).

$$\sum_{e \in \delta^-(v)} \sum_{\omega=0}^{\theta-\tau_e} \gamma_e f_{e,\omega} \geq \sum_{e \in \delta^+(v)} \sum_{\omega=0}^{\theta} f_{e,\omega} \quad \text{for all } v \in V \setminus \{S^+\}, \theta \in \{0, 1, \dots, T-1\}, \quad (1)$$

$$\sum_{e \in \delta^-(v)} \sum_{\omega=0}^{T-\tau_e-1} \gamma_e f_{e,\omega} = \sum_{e \in \delta^+(v)} \sum_{\omega=0}^{T-1} f_{e,\omega} \quad \text{for all } v \in V \setminus \{S^+, S^-\}. \quad (2)$$

According to the choice of  $T$ , no flow remains in the network at time  $T$ , hence  $f_{e,\theta} = 0$  for all  $\theta \geq T - \tau_e$ . Inequality 1 of generalized flow conservation allows holdover of flow, i.e. storage of flow in nodes. If storage at nodes is excluded, then equality holds in inequality 1 for all  $v \in V \setminus \{S^+, S^-\}$ . The flow value  $|f|$  of a GDF  $f$  is the amount of flow sent to the sink within the time horizon  $T$ :

$$|f| = \sum_{e \in \delta^-(S^-)} \sum_{\omega=0}^{T-\tau_e-1} \gamma_e f_{e,\omega}.$$

Analogously, if  $x$  is a generalized static flow, then its flow value is written as  $|x|$ . The arrival pattern of a dynamic flow is a mapping that assigns to every time step  $\theta \in \{0, 1, \dots, T-1\}$  the total amount of flow that arrives at the sink in time steps  $\{0, \dots, \theta\}$ . The GMDF problem is to find a GDF flow of maximum value in a given network  $(G, u, \tau, \gamma, S^+, S^-, T)$ .

A path in a graph  $G$  is a sequence of arcs  $= \{e_1, \dots, e_k\}$  where  $e_k = (v_k, v_{k+1})$ ,  $e_k \in E$ ,  $v_k \in V$ ,  $k \in \mathbb{Z}^+$  and  $v_i \neq v_j$  unless  $i = j$ . A cycle  $C = \{e_1, \dots, e_k, e_1\}$  is a closed path, i.e.  $e_k = (v_k, v_{k+1})$ ,  $v_{k+1} = v_1$ ,  $e_k \in E$ ,  $v_k \in V$ ,  $k \in \mathbb{Z}^+$  and  $v_i \neq v_j$  unless  $i = j$ . The transit time of a path and a cycle is defined as  $\tau_P = \sum_{e \in P} \tau_e$  and  $\tau_C = \sum_{e \in C} \tau_e$  respectively. Analogously, the gain factors can be defined as  $\gamma_P = \prod_{e \in P} \gamma_e$  and  $\gamma_C = \prod_{e \in C} \gamma_e$  for path  $P$  and cycle  $C$ , respectively. Cycles  $C$  with  $\gamma_C = 1$ ,  $\gamma_C > 1$  and  $\gamma_C < 1$  are known as unit gain cycles, flow generalizing cycles and flow absorbing cycles, respectively (see [13]).

For each arc  $e = (v, w) \in E$ , let  $\tilde{e} = (w, v)$  be its reverse arc. Obviously, the reverse of  $\tilde{e} = e$  for all  $e \in E$ . Moreover, let  $G_x = (V, \vec{E} \cup \overleftarrow{E})$  be the residual network with respect to original network  $(G, u, \tau, \gamma, S^+, S^-, T)$  and static flow  $x: E \rightarrow \mathbb{R}_0^+$ . Here  $\vec{E} = \{e | x_e < u_e\}$  is the set of forward arcs having capacity  $u_e - x_e$  and transit time and gain factor  $\tau_e$  and  $\gamma_e$ , respectively, and  $\overleftarrow{E} = \{\tilde{e} | e \in E, x_e > 0\}$ , i.e., set of backward arcs having capacity  $x_e$  for all  $e \in \overleftarrow{E}$  and transit time and gain factor  $\tau_{\tilde{e}} = -\tau_e$  and  $\gamma_{\tilde{e}} = \frac{1}{\gamma_e}$  for all  $e \in E$ .

Using the original network  $(G, u, \tau, \gamma, S^+, S^-, T)$ , the time expanded network  $(G_T = (V_T, E_T), u_T, \gamma_T, S_T^+, S_T^-)$  is defined by copying the network for each time unit (see [8], [14], [13]):

$$V_T = \{v_\theta | v \in V, \theta \in \{0, 1, \dots, T-1\}\},$$

$$E_T = \left\{ e_\theta = (v_\theta, w_{\theta+\tau_e}) \mid e = (v, w) \in E, \theta \in \{0, 1, \dots, T - \tau_e - 1\} \right\}, \text{ for movement arcs,}$$

$$H^T = \{(v_\theta, v_{\theta+1}) \mid v \in V, \theta \in \{0, 1, \dots, T-2\}\}, \text{ for holdover arcs,}$$

$$E_T = E^T \cup H^T.$$

If holdover of flow is not allowed at intermediate nodes, then it is assumed that

$$H^T = \{(v_\theta, v_{\theta+1}) \mid v \in (S^+, S^-), \theta \in \{0, 1, \dots, T-2\}\}.$$

The capacity and gain factors of the arcs  $e \in E_T$  in the time expanded network are defined as follows:

$$u_T^e = \begin{cases} u_e & e = e_\theta \in E^T, \\ \infty & e \in H^T, \end{cases}$$

and

$$\gamma_T^e = \begin{cases} \gamma_e & e = e_\theta \in E^T, \\ 1 & e \in H^T, \end{cases}$$

The source and sink in the time expanded network is  $S_T^+ = S_0^+$  and  $S_T^- = S_{T-1}^-$ .

The well-known result of Ford and Fulkerson ([8], [9]) that dynamic network flows in the static network are in one-to-one correspondence to static flows in the time expanded network has been carried over by [13] to generalized flows.

**Theorem 1.** *Let  $(G, u, \tau, \gamma, S^+, S^-, T)$  be a network with the corresponding time expanded network  $(G_T, u_T, \gamma_T, S_T^+, S_T^-)$ . Then, for a given GDF  $x$  in  $(G, u, \tau, \gamma, S^+, S^-, T)$ , there is a static generalized flow  $x^T$  in  $(G_T, u_T, \gamma_T, S_T^+, S_T^-)$  that sends exactly the same amount of flow to the sink, and vice versa.*

It is well-known that a static  $S_T^+ - S_T^-$  flow can be decomposed into flow along  $S_T^+ - S_T^-$  paths and cycles. Gondran and Minoux [12] proved that a similar decomposition exists for generalized static flows:

**Theorem 2.** *A generalized flow  $x$  in a network  $(G, u, \gamma, S^+, S^-)$  can be decomposed into generalized flows  $x_1, x_2, \dots, x_k$ ,  $k \leq |E|$  with  $x = \sum_{i=1}^k x_i$  such that each generalized flow  $x_i$  is defined on one of the following five structures:*

*Type I: a path from the source  $S^+$  to the sink  $S^-$ ,*

*Type II: a flow generating cycle connected to the sink  $S^-$  by a path,*

*Type III: a path from the source  $S^+$  to a flow absorbing cycle,*

*Type IV: a unit gain cycle,*

*Type V: a flow generating cycle connected to a flow-absorbing cycle by a path.*

An optimality criterion for GMF is given by Onaga ([20], [21]):

**Theorem 3.** *A generalized flow  $x$  in a graph  $G$  with source  $S^+$  and a sink  $S^-$  is a GMF if, in the residual network  $G_x$ , there is neither an  $S^+ - S^-$  path along which flow can be augmented nor a flow generating cycle connected to the sink.*

These results can be applied to generalized dynamic flows, since we can use generalized dynamic flows as static generalized flows in the time-expanded network and vice versa. Thus, the GMDF problem can be solved by using the algorithm for the static GMF problem in a time expanded network. The algorithm has pseudo-polynomial time complexity. Gross and Skutella [13] showed that there is no polynomial time approximation algorithm for the GMDF problem with arbitrary gain factors on the arcs to find a lower bound, unless  $P = NP$ . By reduction from the PARTITION problem, they also proved that for the GMDF problem, there is neither a polynomial algorithm nor a polynomial approximation algorithm on both series-parallel graphs with proportional gains and lossy network where gain factors are  $\leq 1$  with non-proportional gains and transit times, unless  $P = NP$ .

The problem of computing GMF in networks with both gain factors and transit times on arcs is NP-hard and even completely non-approximable. Therefore, [13] considered the special case where the gain factors  $\gamma_e \leq 1$  for all arcs  $e \in E$ , i.e., flow is only lost but never gained along arcs. Such kinds of networks are known as lossy networks. Any networks without flow-generating cycles can be turned into lossy networks by node-dependent scaling of flow values. They assumed that, in each time unit the same percentage of the remaining flow value is lost and they also considered the special case of  $\gamma \equiv 2^{c \cdot \tau}$  for some constant  $c < 0$ . Then, the GMDF problem has been solved on the static network by a variant of the successive shortest path algorithm (SSPA) thus adding one more variant of dynamic flows which can be solved to optimality avoiding the time expanded network.

Onaga ([20], [21]) proved that the iterative use of augmenting flows along highest gain  $S_T^+ - S_T^-$  paths solves the generalized maximum  $S_T^+ - S_T^-$  flow problem. Onaga's algorithm for lossy networks works as follows: Start with the zero flow value and the corresponding residual network. If no  $S_T^+ - S_T^-$  path exists in this residual network, terminate. Otherwise, augment flow along a  $S_T^+ - S_T^-$  path of maximum gain and continue with the resulting flow and residual network. Thus, Onaga's algorithm solves the GMDF problem using the time expanded network at the cost of potentially requiring pseudo-polynomial many augmentations in the pseudo-polynomially large time expanded network.

Gross and Skutella [13] presented an algorithm that is related to Onaga's algorithm in the original network. Before describing their algorithm to solve the special case of  $\gamma_e \leq 1$  for all arcs  $e \in E$  using only the original network, we consider their method of constructing a non-standard version of a time expanded network. For a given network  $(G, u, \tau, \gamma, S^+, S^-, T)$ , let  $\gamma_{v \rightarrow w}$  be the maximum gain of a  $v - w$  path in  $G$  and  $\tau_{v \rightarrow w}$  be the length of a shortest  $v - w$  path in  $G$  where  $\tau_{v \rightarrow w} = \frac{1}{c} \log \gamma_{v \rightarrow w}$ . First, the unique gain network where all the paths from a node  $v \in V$  to a node  $w \in V$  have the same gain can be

constructed as follows. The  $\theta$  -copy of  $G$ , denoted  $\theta G$ ,  $\theta \in \{0, 1, \dots, T - \tau_{S^+ \rightarrow S^-} - 1\}$  is defined by its node and arc set, given by

$$V(\theta G) = \{v_\omega \in V_T \mid v \in V, \omega = \theta + \tau_{S^+ \rightarrow v}\}$$

$$E(\theta G) = \{e_\omega \in E_T \mid e = (v, w) \in E, \omega = \theta + \tau_{S^+ \rightarrow v}\}.$$

If the gain is not uniform, the  $\theta$  -copy of  $G$ ,  $\theta G$ ,  $\theta \in \{0, 1, \dots, T - \tau_{S^+ \rightarrow S^-} - 1\}$  to the following subgraph of the time expanded network  $G_T$  is defined more generally by

$$E(\theta G) = \{e_\omega \in E_T \mid e = (v, w) \in E, \omega = \theta + \tau_{S^+ \rightarrow v}, \omega + \tau_e + \tau_{w \rightarrow S^-} < T\}$$

$$V(\theta G) = \bigcup_{e=(v,w) \in E(\theta G)} \{v, w\}.$$

Similarly, the  $[\theta, \theta']$  copies  $[\theta, \theta']G$  of  $G$ ,  $0 \leq \theta < \theta' \leq T - \tau_{S^+ \rightarrow S^-} - 1$  is given by

$$V([\theta, \theta']G) = \bigcup_{\omega=\theta}^{\theta'} V(\omega G)$$

$$E([\theta, \theta']G) = \bigcup_{\omega=\theta}^{\theta'} E(\omega G) \cup \bigcup_{v \in V} \bigcup_{\omega=\theta+\tau_{S^+ \rightarrow v}}^{\theta'+\tau_{S^+ \rightarrow v}-1} \{(v_\omega, v_{\omega+1})\}.$$

If  $\tau_{S^+ \rightarrow S^-} < T - 1$ , then the network  $\bar{G} = [0, T - \tau_{S^+ \rightarrow S^-} - 1]G$  is defined as the sub-network of  $G_T$  which contains only the nodes and edges of  $G_T$  that can be part of  $S_T^+ - S_T^-$  paths. Otherwise  $\bar{G}$  is an empty graph. It is sufficient to work with  $\bar{G}$  instead of  $G_T$  for our problem.

If an  $S_T^+ - S_T^-$  flow  $f$  in the time expanded network  $G_T$  is considered, it might happen that flow is sent through holdover edges corresponding to the source or sink. In this case, the residual network  $G_T^f$  corresponding to such a flow  $f$  can have reverse holdover edges at source and sink. These reverse holdover edges do not help to construct new  $S_T^+ - S_T^-$  paths in the time expanded network or new flow generating cycle with a path to  $S_T^-$ . According to Theorem 2, they can be removed as well. Let  $\widetilde{G}_T^f$  be the sub-network of  $G_T^f$  that contains no nodes and edges which are not on  $S_T^+ - S_T^-$  paths and no reverse holdover edges at source or sink.

**Example 1.** The evacuation scenario  $G$  presented in fig. 1(a) has capacity and transit time on every arc. The arc  $(s^+, A)$  has 6 units capacity and 2 units transit time; it means, to send 6 units of flow, 2 units time are required. According to the construction of lossy networks, each arc has the same gain factor. By using the SSPA algorithm, we define the

highest gain network  $G'$  as shown in fig. 1(b). Using the relationship between transit time and gain factor, the shortest transit time has the highest gain, so we leave the path having long transit time. The time expanded network  $G_T$  of the evacuation scenario is as shown in fig. 1(c). First, we consider the sub-network  $0G$  and  $1G'$  of time expanded network  $G_T$  as shown in fig. 2(a) and fig. 2(b) that gives the flow pattern of fig. 1(a) at time 0 and fig. 1(b) at time 1, respectively.

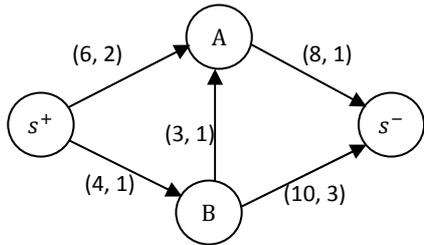


Figure 1(a):  $G$  -evacuation scenario

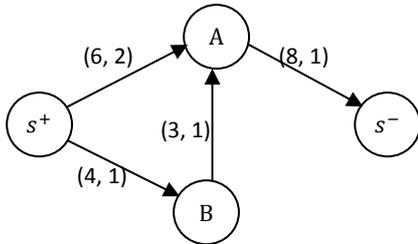


Figure 1(b):  $G'$  -highest gain network of fig. 1(a)

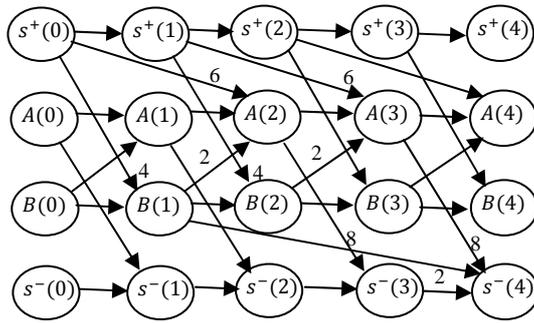


Figure 1(c):  $G_T$  -time expanded network of fig. 1(a)

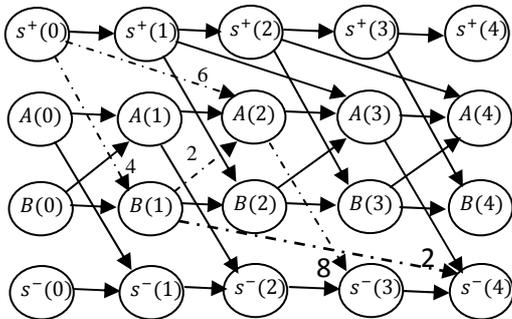


Figure 2(a):  $0G$  -flow at time zero, sub-network of  $G_T$

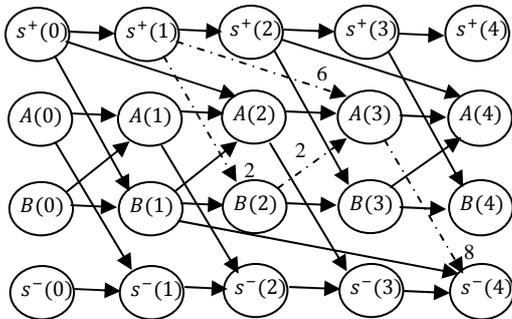


Figure 2(b):  $1G'$  -flow at time one, sub-network of  $G_T$

Now, we consider the sub network  $\bar{G}$  and  $\bar{G}'$  which contains only the nodes and arcs of  $G_T$  that can be the part of source to sink flow as shown in fig. 3(a) for evacuation scenario of fig. 1(a) and fig. 3(b) for highest gain network of fig. 1(b), respectively.

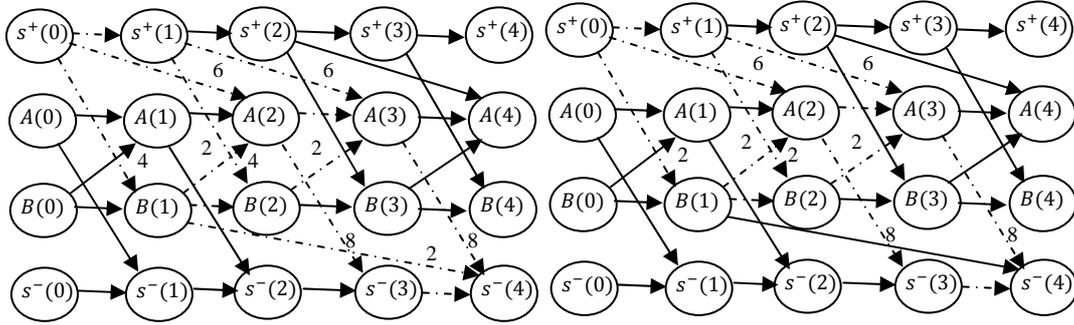


Figure 3(a):  $\bar{G}$  - sub-network of  $G_T$

Figure 3(b):  $\bar{G}'$  -sub-network of  $G_T$

The flow  $x$  in such a unique gain network  $G$  can be defined as the  $\theta$ - flow  $\theta x$  of  $x$  in  $\theta G$  for some  $\theta \in \{0, 1, \dots, T - \tau_{S^+ \rightarrow S^-} - 1\}$  by  $(\theta x)_{e_\omega} = x_e$  for all  $e_\omega \in E(\theta G)$ . Similarly, the  $[\theta, \theta']$  flow of  $[\theta, \theta']x$  in  $[\theta, \theta']G$  for some  $\theta, \theta' \in \{0, 1, \dots, T - \tau_{S^+ \rightarrow S^-} - 1\}$  with  $\theta < \theta'$  by setting for all  $e \in E([\theta, \theta']G)$  has been defined as [13]:

$$([\theta, \theta']x)_e = \begin{cases} 0 & e = (v_\omega, v_{\omega+1}), v \in V \setminus \{S^+, S^-\}, \\ (\theta' - \theta + 1)|x| & e = (S_\omega^+, S_{\omega+1}^+), \omega < \theta, \\ (\theta' - \omega)|x| & e = (S_\omega^+, S_{\omega+1}^+), \theta \leq \omega < \theta', \\ 0 & e = (S_\omega^+, S_{\omega+1}^+), \theta' \leq \omega, \\ 0 & e = (S_\omega^-, S_{\omega+1}^-), \omega \leq \theta + \tau_{S^+ \rightarrow S^-}, \\ (\omega - \theta - \tau_{S^+ \rightarrow S^-} + 1)|x| & e = (S_\omega^-, S_{\omega+1}^-), \theta + \tau_{S^+ \rightarrow S^-} < \omega < \theta' + \tau_{S^+ \rightarrow S^-}, \\ (\theta' - \theta + 1)|x| & e = (S_\omega^-, S_{\omega+1}^-), \omega \geq \theta' + \tau_{S^+ \rightarrow S^-}, \\ x_e & e = e_\omega. \end{cases}$$

Again, set  $\bar{x} = [0, T - \tau_{S^+ \rightarrow S^-} - 1]x$  for a flow  $x$  in  $G$ , if  $\tau_{S^+ \rightarrow S^-} < T - 1$ , and  $\bar{x}$  equal to the zero flow, otherwise. The algorithm of [13] starts with the zero flow, computes a maximum flow in the highest gain, i.e., shortest path subnetwork of the static residual network, augments this flow and repeats this process until no  $S^+ \rightarrow S^-$  path exists in the static residual network. Then, it uses the augmented maximum flows to construct an optimal solution by sending each flow as long as possible through the network - analogous to the temporally repeated flow technique for standard maximum dynamic flow.

**Algorithm 1. [13] Generalized Maximum Dynamic Flow**

Given an instance of the generalized maximum dynamic flow problem  $I = (G, u, \tau, \gamma, S^+, S^-, T)$ .

1. Begin with  $i = 0$  and the static zero flow  $x_0 \equiv 0$ .
2. If no  $S^+ \rightarrow S^-$  path exists in  $G_{x_i}$ , or if  $\tau_{S^+ \rightarrow S^-} \geq T$  in  $G_{x_i}$ , then set  $k = i - 1$  and go to step 6.

3. Restrict the static residual network  $G_{x_i}$  to the network  $G'_{x_i}$  containing only paths of maximum gain and compute a generalized maximum flow  $x'_i$  in  $G'_{x_i}$ .
4. Define  $x_{i+1}$  by adding  $x'_i$  to  $x_i$  as follows:  $(x_{i+1})_e = (x_i)_e + (x'_i)_e + \gamma_e^{-1}(x'_i)_{\bar{e}}$  for all  $e \in E$ . (Notice that  $x_{i+1}$  is a feasible flow in  $G$ , since  $x'_i$  is a feasible flow in a restricted residual network of  $x_i$ .)
5. Set  $i = i + 1$  and go to step 2.
6. Construct the generalized maximum dynamic flow  $f = \sum_{j=0}^k \bar{x}'_j$ .

Gross and Skutella [13] proved the correctness of Algorithm 1 with the following result.

**Theorem 4.** *Algorithm 1 computes the generalized maximum dynamic flow.*

The running time of Algorithm 1 is  $O(|V||E| \log_{\frac{|E|}{|V|\log|V|}} |V|.T)$ , where the highest gain path can be found in  $O(|V||E|)$  time and the generalized maximum flow along the highest gain path can be found using standard maximum flow algorithms. There are at most  $T$  iterations with running time dominated by the maximum flow computations having running time  $O(|V||E| \log_{\frac{|E|}{|V|\log|V|}} |V|.T)$  where  $O(|V||E| \log_{\frac{|E|}{|V|\log|V|}} |V|)$  gives the maximum flow. For details see [13] where they proved that there is always an optimal solution that uses no holdover at any node except for  $S^+$  and  $S^-$ . Furthermore, they proved that all such optimal solutions allocate the same arrival pattern presenting following theorem.

**Theorem 5.** *Let  $I = (G, u, \tau, \gamma, S^+, S^-, T)$  be an instance of the GMDF problem such that  $\gamma \equiv 2^{c.\tau}$  for some constant  $c < 0$ . Let  $f$  and  $f'$  be generalized maximum dynamic flows for  $I$  that do not use holdover in any node except for  $S^+$  and  $S^-$ . Then  $f$  and  $f'$  have the same arrival pattern.*

### 3. Generalized Dynamic Contraflow Problems on Lossy Network

In this section, we present a pseudo-polynomial time algorithm for solving the GMDCF problem on a single source and a single sink lossy network (c.f. Subsection 3.1). Then, we proved that the GMDCF algorithm also solves the GEACF problem on a single source and a single sink lossy network (c.f. Subsection 3.2).

### 3.1 Generalized Maximum Dynamic Contraflow Problem

The computational complexity of contraflow problems has been studied by Arulselvan [1] and Rebennack et al. [24]. Given a network  $(G, u, \tau, \gamma, S^+, S^-, T)$  with a single source  $S^+$  and a single sink  $S^-$  having travel time  $\tau_{vw} \in \mathbb{Z}^+$  for each arc  $e = (v, w) \in E$  with  $\tau_{vw} = \tau_{wv}$  for  $(v, w), (w, v) \in E$ , capacity  $u_{vw} \in \mathbb{Z}^+$  for each  $(v, w) \in E$ , and gain factor  $\gamma_{vw} \in \mathbb{R}^+$  with  $\gamma_{vw} = \gamma_{wv}$  for  $(v, w), (w, v) \in E$ , the GMDCF problem requires to find the maximum amount of flow that can be sent within the given integer time  $T$  units from the source  $S^+$  to the sink  $S^-$  if the direction of the arcs can be reversed at time 0.

Strongly polynomial time algorithms for the MDCF problem and the static MCF problem have been presented in ([1], [24]). Time complexities of these algorithms for the MDCF and MCF are  $O(K_2|V|, |E|) + K_3(|V|, |E|)$  and  $O((K_1|V|, |E|) + K_2(|V|, |E|))$ , respectively, where  $K_1(|V|, |E|)$ ,  $K_2(|V|, |E|)$ , and  $K_3(|V|, |E|)$  are the worst case complexity for an algorithm to solve the maximum flow problem, the flow decomposition problem, and the minimum cost flow problem, respectively (e.g.  $K_1(|V|, |E|) = O(|V|^2 \cdot \sqrt{E})$ ,  $K_2(|V|, |E|) = O(|V| \cdot |E|)$ , and  $K_3(|V|, |E|) = O(|V|^2 \cdot |E|^3 \cdot \log |V|)$ ).

The authors also prove that the quickest contraflow problem with single-source and single-sink is solvable in polynomial time. By reductions from 3-SAT and from PARTITION, respectively, it is proved that the MDCF problem remains NP-hard in the strong sense even with two sources and one sink or vice versa ([18], [24]).

We modify the MDCF algorithm of [24] making use of the GMDF Algorithm 1. Then, we obtained a GMDCF solution (c.f. Algorithm 2) on network  $(G, u, \tau, \gamma, S^+, S^-, T)$ , which has the earliest arrival property (c.f. Subsection 3.2).

Algorithm 2 modifies the algorithm of [24] for the MDCF problem on general graphs replacing Step 2 by Algorithm 1 for GMDF on lossy network  $(G, u, \tau, \gamma, S^+, S^-, T)$ . Our algorithm solves the GMDCF problem for single source and single sink lossy network  $(G, u, \tau, \gamma, S^+, S^-, T)$  together with the GEACF.

**Algorithm 2.** *Generalized Maximum Dynamic Contraflow on Lossy Network*

1. Construct the transformed network  $(G', u', \tau', \gamma', S^+, S^-, T)$  where graph  $G' = (V, E')$  with arc set defined as  $e = (v, w) \in E'$  if  $(v, w) \in E$  or  $(w, v) \in E$  with highest gain factor  $\gamma_e$ .

The arc capacity function  $u'$  for each arc  $e = (v, w) \in E'$  is given by  $u'_{vw} = u_{vw} + u_{wv}$

For all arcs  $e = (v, w) \in E'$  the transit time is

$$\tau'_{vw}(= \tau'_{wv}) = \begin{cases} \tau_{vw} & \text{if } (v, w) \in E \\ \tau_{wv} & \text{otherwise} \end{cases}$$

and gain factor is  $\gamma'_{vw}$  is  $\gamma'_{vw}(= \gamma'_{vw}) \equiv 2^{c\tau'_{vw}}$ ,  $c < 0$ .

2. In the transformed network  $(G', u', \tau', \gamma', S^+, S^-, T)$ , generate a generalized dynamic temporally repeated flow with capacity  $u'$ , travel time  $\tau'_{vw}$  and the unique gain factor  $\gamma'_{vw}$  using Algorithm 1.
3. Obtain flow decomposition into path and cycle flows of the resulting network from Step 2. Remove the cycle flows.
4. Arc  $(w, v) \in E$  is reversed, if and only if the flow along arc  $(v, w)$  is greater than  $u_{vw}$  or if there is a non-negative flow along arc  $(v, w) \notin E$  with highest gain and the resulting flow is GMDF with the arc reversal for the network  $(G, u, \tau, \gamma, S^+, S^-, T)$ .

Before we prove the correctness of Algorithm 2, we give the proof of the following Lemma 1 that interprets the relationship between the GMDCF for lossy network having single source and single sink and the GMCF for the corresponding time expanded network.

**Lemma 1.** *For single source and single sink lossy network  $(G, u, \tau, \gamma, S^+, S^-, T)$ , the generalized maximum amount of flow in the GMDCF problem is less than or equal to the optimal flow in the GMCF problem for the corresponding time expanded network  $(G_T, u_T, \gamma_T, S_T^+, S_T^-)$ .*

**Proof:** According to Ford and Fulkerson ([8], [9]), a dynamic network flow of a network is equivalent to the static flow in the time expanded graph of corresponding network and vice versa. Based on this result, [13] proved that for a given generalized dynamic flow  $x$  in  $(G, u, \tau, \gamma, S^+, S^-, T)$ , there is a static generalized flow  $x^T$  in  $(G_T, u_T, \gamma_T, S_T^+, S_T^-)$  that sends exactly the same amount of flow to the sink, and vice versa according to Theorem 1.

Similarly, [24] proved that for a general graph, the maximum amount of flow in the MDCF problem is less than or equal to the optimal flow in the MCF problem for the corresponding time expanded graph. From these result, we directly conclude that every feasible flow in the GMDCF problem has an equivalent feasible flow to the GMCF in the corresponding time expanded graph. ■

**Theorem 6.** *Algorithm 2 solves the GMDCF problem for lossy network  $(G, u, \tau, \gamma, S^+, S^-, T)$  to optimality.*

**Proof:** We prove this theorem in two steps. First, we give the feasibility of the Algorithm 2 and then prove the correctness. Since, Steps 1-3 are feasible, it is suffices to show that Step 4 is feasible. After generalized flow decomposition, the optimal solution outputs a set of paths from source to sink and a set of cycles with positive flows obtained from Step

3. The positive flow along all cycles can be cancelled and hence, there is no flow along any cycle. Therefore, there is either a flow along arc  $(v, w)$  or  $(w, v)$  but never in both arcs. This proves that the flow is not greater than the reversed capacities on all the arcs at all time units.

Next, we show the optimality. Due to the feasibility, we conclude that every feasible GMDF of the dynamic flow problem in the transformed network  $(G', u', \tau', \gamma', S^+, S^-, T)$  is feasible to the GMDCF in the original network  $(G, u, \tau, \gamma, S^+, S^-, T)$ , i.e.,

$$[(G, u, \tau, \gamma, S^+, S^-, T)]_{GMDCF\ opt} \geq [(G', u', \tau', \gamma', S^+, S^-, T)]_{GMDF\ opt}. \quad (3)$$

According to Lemma 1, we have

$$[G, u, \tau, \gamma, S^+, S^-, T]_{GMDCF\ opt} \leq [G_T, u_T, \gamma_T, S_T^+, S_T^-]_{GMCF\ opt}.$$

GMCF in network  $(G_T, u_T, \gamma_T, S_T^+, S_T^-)$  is equivalent to GMF in the transformed network  $(G'_T, u'_T, \gamma'_T, S_T^+, S_T^-)$ , where graph  $G'_T = (V_T, E'_T)$  with arc set  $E'_T$  defined as  $(v, w) \in E'_T$ , if  $(v, w) \in E_T$  or  $(w, v) \in E_T$ , the capacity function  $u'_T$  is given by  $u'_{vw} = u_{vw} + u_{wv}$  and the unique gain  $\gamma'_T$  is given by  $\gamma'_{vw} \equiv 2^{c\tau_{vw}}$ , for  $c < 0$  for all  $e = (v, w) \in E'_T$ . Thus, we have

$$[G_T, u_T, \gamma_T, S_T^+, S_T^-]_{GMCF\ opt} = [(G'_T, u'_T, \gamma'_T, S_T^+, S_T^-)]_{GMF\ opt}.$$

According to the Ford and Fulkerson [8] and the minimum cost circulation flow (MCCF) algorithm, the maximum flow in the time expanded graph  $(G'_T, u'_T, \gamma'_T, S_T^+, S_T^-)$ , can be obtained by a temporally repeated chain flow of the transformed network  $(G', u', \tau', \gamma', S^+, S^-, T)$ . This implies that

$$\begin{aligned} [(G, u, \tau, \gamma, S^+, S^-, T)]_{GMDCF\ opt} &= (G'_T, u'_T, \gamma'_T, S_T^+, S_T^-)_{GMF\ opt} \\ &= [(G', u', \tau', \gamma', S^+, S^-, T)]_{GMDF\ opt}. \end{aligned}$$

The proof is finished by combining Inequality 3 with the preceding result. ■

**Theorem 7.** Algorithm 2 solves the GMDCF problem in pseudo-polynomial time for single source single sink lossy network.

**Proof:** Step 1 and Step 4 are solved in linear time. The flow decomposition in Step 3 can be done in  $O(|V||E'|)$  time. The running time of the Algorithm 1 is  $O(|V||E'| \log \frac{|E'|}{|V|\log|V|} |V|.T)$  [13]. Hence, the total complexity of the Algorithm 2 is

$$O\left(|V||E'| + O(|V||E'| \log \frac{|E'|}{|V|\log|V|} |V|.T)\right). \quad \blacksquare$$

**Example 2.** Consider the evacuation scenario as shown in fig. 4(a), with two-way arcs each showing capacity and transit time. Every arc has equal gain factor. First, transform the network using Algorithm 2 as shown in fig. 4(b). Then, using Algorithm 1, we obtain highest gain path and GMF as shown in fig. 4(c) to 4(e). The GMF obtained from the fig. 4(e), as the temporally repeated flow, gives the GMDCF of the original network fig. 4(a).

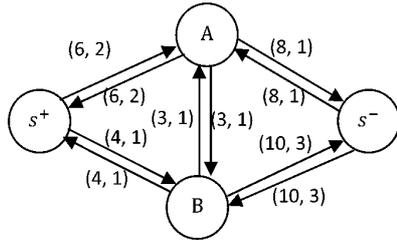


Figure 4(a):  $G$ - evacuation scenario

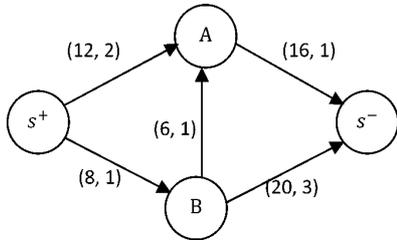


Figure 4(b):  $G'$ -contraflow of fig. 4(a)

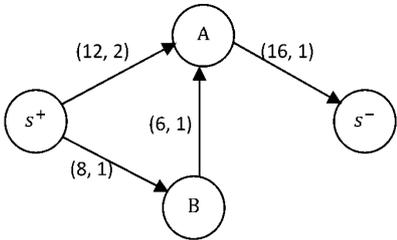


Figure 4(c):  $G''$ - highest gain network of fig. 4(b)

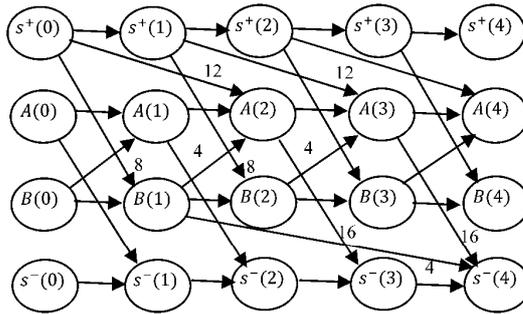


Figure 4(d):  $G_T'$ -time expanded network

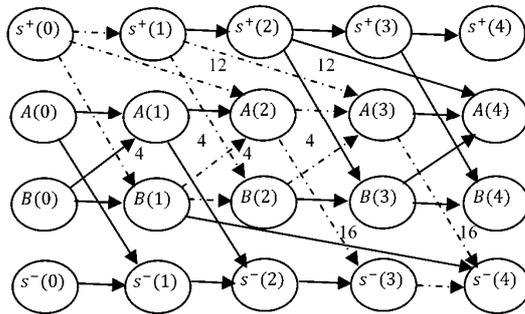


Figure 4(e):  $\overline{G''}$ -sub-network of  $G_T'$

### 3.2 Generalized Earliest Arrival Contraflow Problem

Dhamala and Pyakurel [7] presented a strongly polynomial time algorithm for EACF problem on single source and single sink series-parallel graph. The time complexity of their algorithm is  $O(|V||E'| + |E'|\log|E'|)$ . This is the first mathematical model for EACF problem. The GEACF problem for single source and single sink lossy network is defined as follows.

Given a lossy network  $(G, u, \tau, \gamma, S^+, S^-, T)$  with single-source  $S^+ \in V$  and single-sink  $S^- \in V$  having travel time  $\tau_{vw} \in \mathbb{Z}^+$  on each arc  $e = (v, w) \in E$  with  $\tau_{vw} = \tau_{wv}$  if  $(v, w), (w, v) \in E$  and capacity  $u_{vw} \in \mathbb{Z}^+$  for each  $(v, w) \in E$ , and gain factors  $\gamma_{vw} \in \mathbb{R}^+$  on each arc  $e = (v, w) \in E$  with  $\gamma_{vw} = \gamma_{wv}$  for each arc  $(v, w), (w, v) \in E$ , the GEACF problem on a single source and a single sink lossy network requires the maximum amount of flow that can be sent in every time period  $\theta, 0 \leq \theta \leq T$ , from the source to the sink if the arc can be reversed at time zero.

In Subsection 3.1, we have seen that there is always an optimal solution for GMDCF problem on lossy network. Here, we show that the optimal solution induced by Algorithm 2 has the earliest arrival property. Thus, we will solve the GEACF problem for lossy network.

**Theorem 8** *Any GMDCF solution induced by Algorithm 2 has the earliest arrival property for lossy network.*

**Proof:** Algorithm 2 gives a GMDCF solution on the modified network  $(G', u', \tau', \gamma', S^+, S^-, T)$ . The fact that the temporarily repeated GMDCF flow obtained by Algorithm 2 has the earliest arrival flow property follows from Theorem 5 of [13]. It has been proved by contradiction that the flow obtained from Algorithm 2 does not have the same arrival pattern. They decomposed the generalized flows into five types of elementary generalized flow according to Theorem 2 of [12] and proved the contradiction. ■

**Theorem 9** *Algorithm 2 solves the GEACF problem optimally in pseudo-polynomial time for lossy network.*

According to the proofs of Theorem 6- Theorem 8 and the GMDCF Algorithm 2 the proof of Theorem 9 follows directly, because the algorithm gives the temporarily repeated flow using the SSPA. From the definition, all the GEACF solutions are the GMDCF solutions. Theorem 6 and Theorem 8 prove that all the GMDCF has the earliest arrival property for a single source and a single sink lossy network. Therefore, we state the Corollary 1.

**Corollary 1.** *Any GMDCF solution is also an GEACF solutions and vice versa for lossy network.*

## 4. Conclusions

Gross and Skutella [13] have presented a pseudo-polynomial algorithm for the GMDF problem for lossy network. Arulselvan [1] and Rebennack et al. [24] have presented a strongly polynomial time algorithm that gives the MDCF solution with single source and single sink for general graphs. Based on their algorithms, we presented a pseudo-polynomial time algorithm for lossy network with single source and single sink that

computes the GMDCF solution within the time horizon  $T$  in a discrete time setting. The presented algorithm also gives the GEACF solution because it yields the existence of temporally repeated flow on lossy network which has the earliest arrival property.

To the best of our knowledge, this paper is first to introduce a mathematical model of the GMDCF problem as well as the GEACF problem in evacuation planning.

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